ON SOME HADAMARD-TYPE INEQUALITIES FOR DIFFERENTIABLE $m-$CONVEX FUNCTIONS

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ABSTRACT. In this paper some new inequalities are proved related to left hand side of Hermite-Hadamard inequality for the classes of functions whose derivatives of absolute values are $m-$convex. New bounds and estimations are obtained. Applications for some Theorems are given as well.

1. INTRODUCTION

Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well-known in the literature as Hermite-Hadamard inequality, holds [see [5], p. 137];

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results, generalizations and new inequalities related to the inequality presented above see [1]-[4].

In [10], Toader defined the concept of $m-$convexity as the following;

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0, b]$ for which $f(0) \le 0$.

Several papers have been written on $m-$convex functions on $[0, b]$ and we refer the papers [7], [8], [9], [10], [11], [12], [13], [14], [15] and [16]. In [17], Dragomir and Agarwal proved following inequality for convex functions;

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I^0 and $a, b \in I$, where $a < b$. If $|f'|^q$ is convex on $[a, b]$, then the following inequality holds;*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \tag{1.2}$$

In [4], Pearce and Pečarić proved the following inequalities for convex functions;

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Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I^0 and $a, b \in I$, where $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} \tag{1.3}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{1.4}$$

In [7], Bakula *et al.* proved the following inequality for $m-$convex functions;

Theorem 3. *Let I be an open real interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L[a, b]$, where $0 \leq a < b < \infty$. If $|f'|^q$ is $m-$convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \in [1, \infty)$, then;*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq\ & \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \left(\frac{m\left|f'\left(\frac{a}{m}\right)\right|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

In [13], Dragomir established following inequalities of Hadamard-type similar to above.

Theorem 4. *Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then one has the inequality:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2}dx \\ &\leq \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2} + m\frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{1.5}$$

The following classical inequality is well-known in the literature as Favard's inequality (see [18], [19, p.216]);

Theorem 5. *(i) (Favard's inequality) Let f be a non-negative concave function on $[a, b]$. If $q \geq 1$, then*

$$\frac{2^q}{q+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^q \geq \frac{1}{b-a}\int_a^b f^q(x)dx. \tag{1.6}$$

If $0 < q < 1$ the reverse inequality holds in (1.6).

(ii) (Thunsdorff's inequality) If f is a non-negative, convex function with $f(a) = 0$, then for $q \geq 1$ the reversed inequality holds in (1.6).

Motivated by the above results, in this paper we consider new Hadamard-type inequalities for functions whose derivatives of absolute values are $m-$convex by using fairly elementary analysis and some classical inequalities like Hölder inequality, Power-mean inequality and Favard's inequality. These new results gives new upper bounds for the Theorem 2-3. We also give some applications.

2. MAIN RESULTS

To prove our main results, we use following Lemma which was used by Alomari *et al.* (see [6]).

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I where $a, b \in I$, with $a < b$. Let $f' \in L[a,b]$, then the following equality holds;*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \\
= & \frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)\,a\right)dt + \int_0^1 (t-1)\,f'\left(tb + (1-t)\,\frac{a+b}{2}\right)dt\right].
\end{aligned}
$$

Theorem 6. *Let $f : [0,\infty) \to \mathbb{R}$, be a differentiable mapping such that $f' \in L[a,b]$. If $|f'|$ is $m-$convex on $[a,b]$, where $0 \le a < b < \infty$ and for some fixed $m \in (0,1]$, then the following inequality holds;*

$$
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \min\{T_1, T_2, T_3, T_4\} \tag{2.1}
$$

where

$$
\begin{aligned}
T_1 &= \frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + m\left[\frac{\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|}{2}\right]\right], \\
T_2 &= \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + m\left|f'\left(\frac{a+b}{2m}\right)\right| + \frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right], \\
T_3 &= \frac{b-a}{12}\left[\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2} + 2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right], \\
T_4 &= \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + m\left|f'\left(\frac{a+b}{2m}\right)\right| + \frac{m\left|f'\left(\frac{a}{m}\right)\right| + \left|f'(b)\right|}{2}\right].
\end{aligned}
$$

Proof. From the equality which is given in the Lemma 1 and by using the properties of modulus, we have

$$
\begin{aligned}
& \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\le & \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right|dt\right. \\
& \left. + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right|dt\right].
\end{aligned} \tag{2.2}
$$

By using $m-$convexity of $|f'|$ on $[a,b]$, we know that for any $t \in [0,1]$

$$
\left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right| \le t\left|f'\left(\frac{a+b}{2}\right)\right| + m\,(1-t)\left|f'\left(\frac{a}{m}\right)\right| \tag{2.3}
$$

and

$$\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right| \leq (1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + mt\left|f'\left(\frac{b}{m}\right)\right|. \tag{2.4}$$

From the inequalities (2.3) and (2.4), we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 t\left(t\left|f'\left(\frac{a+b}{2}\right)\right| + m(1-t)\left|f'\left(\frac{a}{m}\right)\right|\right)dt\right. \\
&\left. + \int_0^1 (1-t)\left((1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + mt\left|f'\left(\frac{b}{m}\right)\right|\right)dt\right].
\end{aligned}$$

By calculating the above integrals, we get the following inequality;

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + m\left[\frac{\left|f'\left(\frac{a}{m}\right)\right| + \left|f'\left(\frac{b}{m}\right)\right|}{2}\right]\right]. \tag{2.5}$$

Analogously, we obtain the following inequalities;

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + m\left|f'\left(\frac{a+b}{2m}\right)\right| + \frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right] \tag{2.6}$$

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2} + 2m\left|f'\left(\frac{a+b}{2m}\right)\right|\right] \tag{2.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + m\left|f'\left(\frac{a+b}{2m}\right)\right| + \frac{m\left|f'\left(\frac{a}{m}\right)\right| + \left|f'(b)\right|}{2}\right]. \tag{2.8}$$

From the inequalities (2.5), (2.6), (2.7) and (2.8), we get the desired result. □

Corollary 1. *If we choose $m = 1$ in (2.1), we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{\left|f'(a)\right| + \left|f'(b)\right|}{2}\right].$$

Corollary 2. *Under the assumptions of Theorem 6;*

i) If we choose $m = 1$ and $|f'|$ is increasing in (2.1), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'(b)\right|\right].$$

ii) If we choose $m=1$ and $|f'|$ is decreasing in (2.1), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)\right|\right].$$

iii) If we choose $m=1$ and $\left|f'\left(\frac{a+b}{2}\right)\right|=0$ in (2.1), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right].$$

iv) If we choose $m=1$ and $|f'(a)|=|f'(b)|=0$ in (2.1), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{6}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Theorem 7. *Let $f:[0,\infty)\to\mathbb{R}$, be a differentiable mapping such that $f'\in L[a,b]$. If $|f'|^{\frac{p}{p-1}}$ is $m-$convex on $[a,b]$, where $0\leq a<b<\infty$, for some fixed $m\in(0,1]$ and $p>1$, then the following inequality holds;*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\min\{U_1,U_2,U_3,U_4\} \\ &\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\min\{U_1,U_2,U_3,U_4\}\end{aligned} \tag{2.9}$$

where $\frac{1}{q}+\frac{1}{p}=1$ and

$$\begin{aligned}
U_1 &= \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \\
U_2 &= \left(\left|f'(a)\right|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \\
U_3 &= \left(\left|f'(a)\right|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(b)\right|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}, \\
U_4 &= \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1 and by using the properties of modulus, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ &\frac{b-a}{4}\left[\int_0^1|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1|t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right].\end{aligned} \tag{2.10}$$

By applying the Hölder inequality to the inequality (2.10), we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is easy to see that

$$\int_0^1 t^p dt=\int_0^1 (1-t)^p dt=\frac{1}{p+1}.$$

Hence, by $m-$convexity of $|f'|^q$ on $[a,b]$, we obtain the inequality;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.11}$$

By a similar argument to the proof of Theorem 6, analogously, we obtain the following inequalities;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(|f'(a)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(|f'(a)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(|f'(b)|^q+m\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{2.13}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.14}$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

From the inequalities (2.11)-(2.14), we obtain the inequality in (2.9). The second inequality in (2.9) follows from facts that;

$$\lim_{p\to\infty}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}=1 \quad , \quad \lim_{p\to 1^+}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}=\frac{1}{2}$$

and

$$\frac{1}{2}<\left(\frac{1}{1+p}\right)^{\frac{1}{p}}<1.$$

□

Corollary 3. *Under the assumptions of Theorem 7, if we choose $m=1$, we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 4. *Under the assumptions of Theorem 7;*

i) If we choose $m=1$ and $|f'|^{\frac{p}{p-1}}$ is increasing in (2.9), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

ii) If we choose $m=1$ and $|f'|^{\frac{p}{p-1}}$ is decreasing in (2.9), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}.$$

iii) If we choose $m=1$ and $\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}=0$ in (2.9), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

iv) If we choose $m=1$ *and* $|f'(a)|^{\frac{p}{p-1}}=|f'(b)|^{\frac{p}{p-1}}=0$ *in (2.9), we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Theorem 8. *Let* $f:[0,\infty)\to\mathbb{R}$ *, be a differentiable mapping such that* $f'\in L[a,b]$. *If* $|f'|^q$ *is* $m-$*convex on* $[a,b]$, *where* $0\leq a<b<\infty$, *for some fixed* $m\in(0,1]$ *and* $q\geq 1$, *then the following inequality holds;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\min\{V_1,V_2,V_3,V_4\} \tag{2.15}$$

where

$$\begin{aligned}
V_1 &= \left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}},\\
V_2 &= \left(\frac{1}{6}|f'(a)|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}},\\
V_3 &= \left(\frac{1}{6}|f'(a)|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{6}|f'(b)|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}},\\
V_4 &= \left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{6}|f'(b)|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1, we can write

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\ &\frac{b-a}{4}\left[\int_0^1|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1|t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right].
\end{aligned}$$

By applying the Power-mean inequality, we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\ &\frac{b-a}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Now by using $m-$convexity of $|f'|^q$ on $[a,b]$ and by computing the integrals, we obtain the following inequality;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.16}$$

Hence, by a similar argument to the proofs of Theorem 6-7, analogously, we obtain the following inequalities;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{6}\left|f'(a)\right|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{6}\left|f'(a)\right|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{1}{6}\left|f'(b)\right|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{6}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{1}{6}\left|f'(b)\right|^q+\frac{m}{3}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.19}$$

By the inequalities (2.16)-(2.19), we obtain the inequality (2.15). □

Corollary 5. *Under the assumptions of Theorem 8, if we choose $m=1$, we obtain the inequality;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{6}\left|f'(b)\right|^q+\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 6. *Under the assumptions of Theorem 8;*

i) If we choose $m=1$ and $|f'|^q$ is increasing in (2.15), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

ii) If we choose $m=1$ and $|f'|^q$ is decreasing in (2.15), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(a)\right|^q\right)^{\frac{1}{q}}.$$

iii) If we choose $m=1$ and $\left|f'\left(\frac{a+b}{2}\right)\right|^q=0$ in (2.15), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

iv) If we choose $m=1$ and $|f'(a)|^q=|f'(b)|^q=0$ in (2.15), we obtain the inequality;

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Theorem 9. *Let $f,g:[0,b]\to\mathbb{R}$, be concave and $m-$concave functions, $m\in(0,1]$, where $0\leq a<b<\infty$ and $q\geq 1$. Then*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\ \geq\ & \frac{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}{16}\\ &\times\left(\frac{1}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]\left[g(x)+mg\left(\frac{x}{m}\right)\right]dx\right).\end{aligned}$$

where $\frac{1}{q}+\frac{1}{p}=1$.

If f,g are convex and $m-$convex functions, with $f(0)=0$, then the reverse of the above inequality holds.

Proof. Since f,g are $m-$concave, by using the inequality (1.5), we can write

$$f\left(\frac{a+b}{2}\right)\geq\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx$$

and

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx.$$

By using Favard's inequality for $p-$th powers of both sides of inequality, we have

$$\begin{aligned} f^p\left(\frac{a+b}{2}\right) &\geq \left(\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\right)^p \\ &\geq \frac{p+1}{2^p}\left[\frac{1}{b-a}\int_a^b \left(\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}\right)^p dx\right] \end{aligned}$$

and similarly, we have

$$g^q\left(\frac{a+b}{2}\right) \geq \frac{q+1}{2^q}\left[\frac{1}{b-a}\int_a^b \left(\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}\right)^q dx\right].$$

It follows that

$$f\left(\frac{a+b}{2}\right) \geq \frac{(p+1)^{\frac{1}{p}}}{2}\left[\frac{1}{b-a}\int_a^b \left(\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}\right)^p dx\right]^{\frac{1}{p}}$$

and

$$g\left(\frac{a+b}{2}\right) \geq \frac{(q+1)^{\frac{1}{q}}}{2}\left[\frac{1}{b-a}\int_a^b \left(\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}\right)^q dx\right]^{\frac{1}{q}}.$$

By multiplying both sides of the above inequalities, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\geq \frac{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}{4}\left(\frac{1}{b-a}\int_a^b \left(\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}\right)^p dx\right)^{\frac{1}{p}} \\ &\quad \times \left(\frac{1}{b-a}\int_a^b \left(\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}\right)^q dx\right)^{\frac{1}{q}}. \end{aligned}$$

By using Hölder inequality, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\geq \frac{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}{16} \\ &\quad \times \left(\frac{1}{b-a}\int_a^b \left[f(x)+mf\left(\frac{x}{m}\right)\right]\left[g(x)+mg\left(\frac{x}{m}\right)\right]dx\right). \end{aligned}$$

If f, g are $m-$convex, then using Thunsdorff's inequality we obtain desired result. □

Corollary 7. *Under the assumptions of Theorem 9, if we choose $m=1$, we obtain the inequality;*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\geq \frac{(p+1)^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}{4} \\ &\times\left(\frac{1}{b-a}\int_a^b f(x)g(x)dx\right). \end{aligned}$$

3. APPLICATIONS TO SOME SPECIAL MEANS

We now consider the applications of our Theorems to the following special means

a) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0,$$

b) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if} \quad a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if} \quad a\neq b \end{cases},\quad a,b\geq 0,$$

c) The $p-$logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \quad a\neq b \\ a & \text{if} \quad a=b \end{cases},\quad p\in\mathbb{R}\backslash\{-1,0\};\quad a,b>0.$$

We now derive some sophisticated bounds of the above means.

Proposition 1. *Let $a,b\in\mathbb{R}$, $0<a<b$ and $n\in\mathbb{Z}$, $|n|\geq 2$. Then, we have:*

$$|A^n(a,b)-L_n^n(a,b)|\leq \min\{K_1,K_2,K_3,K_4\}$$

where

$$\begin{aligned} K_1 &= \frac{n(b-a)}{12}\left[2|A(a,b)|^{n-1}+m\left[A\left(\left|\left(\frac{a}{m}\right)\right|^{n-1},\left|\left(\frac{b}{m}\right)\right|^{n-1}\right)\right]\right], \\ K_2 &= \frac{n(b-a)}{12}\left[|A(a,b)|^{n-1}+m\left|\frac{A(a,b)}{m}\right|^{n-1}+A\left(|a|^{n-1},m\left|\frac{b}{m}\right|^{n-1}\right)\right], \\ K_3 &= \frac{n(b-a)}{12}\left[A\left(|a|^{n-1}+|b|^{n-1}\right)+2m\left|\frac{A(a,b)}{m}\right|^{n-1}\right], \\ K_4 &= \frac{n(b-a)}{12}\left[|A(a,b)|^{n-1}+m\left|\frac{A(a,b)}{m}\right|^{n-1}+A\left(m\left|\frac{a}{m}\right|^{n-1},|b|^{n-1}\right)\right]. \end{aligned}$$

Proof. The proof is immediate from Theorem 6 applied for $f(x)=x^n$, which is an $m-$convex function. □

Proposition 2. *Let $a,b\in\mathbb{R}$, $0<a<b$ and $n\in\mathbb{Z}$, $|n|\geq 2$, $k\geq 1$. Then, we have:*

$$\left|A^{\frac{n}{k}}(a,b)-L_n^{\frac{n}{k}}(a,b)\right|\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\min\{L_1,L_2,L_3,L_4\}$$

where

$$L_1 = \frac{n}{k} 2A \left[\left(2A \left(\frac{1}{3} \left| (A(a,b)) \right|^{\frac{q(n-k)}{k}}, \frac{m}{6} \left| \frac{a}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} + \left(2A \left(\frac{1}{3} \left| (A(a,b)) \right|^{\frac{q(n-k)}{k}}, \frac{m}{6} \left| \frac{b}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} \right],$$

$$L_2 = \frac{n}{k} 2A \left[\left(2A \left(\frac{1}{6} |a|^{\frac{q(n-k)}{k}} + \frac{m}{3} \left| \frac{A(a,b)}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} + \left(2A \left(\frac{1}{3} \left| (A(a,b)) \right|^{\frac{q(n-k)}{k}}, \frac{m}{6} \left| \frac{b}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} \right],$$

$$L_3 = \frac{n}{k} 2A \left[\left(2A \left(\frac{1}{6} |a|^{\frac{q(n-k)}{k}} + \frac{m}{3} \left| \frac{A(a,b)}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} + \left(2A \left(\frac{1}{6} |b|^{\frac{q(n-k)}{k}} + \frac{m}{3} \left| \frac{A(a,b)}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} \right],$$

$$L_4 = \frac{n}{k} 2A \left[\left(2A \left(\frac{1}{3} \left| (A(a,b)) \right|^{\frac{q(n-k)}{k}}, \frac{m}{6} \left| \frac{a}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} + \left(2A \left(\frac{1}{6} |b|^{\frac{q(n-k)}{k}} + \frac{m}{3} \left| \frac{A(a,b)}{m} \right|^{\frac{q(n-k)}{k}} \right) \right)^{\frac{1}{q}} \right].$$

Proof. The assertion follows from Theorem 8 applied to $f(x) = x^{\frac{n}{k}}$, which is an $m-$convex function. □

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